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ABOUT THE SUM OF QUASICONVEX FUNCTIONS

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Abstract. We provide conditions under which the sum of two quasiconvex functions is also a quasiconvex function.

1. Introduction and Preliminaries

In utility theory [10] is well-known the result: "if u is a quasiconcave real function of the form

$$u(x_1, x_2, \dots, x_n) = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n),$$

where $f_1, f_2, \dots, f_n$ are real continuous functions whose domains are intervals on the real line, then at least $n-1$ of the functions f_i, $i = 1, 2, \dots, n$ must be concave functions".
In optimization theory, theorems of the alternative (transposition theorems) have proved to be an important tool to derive existence of Lagrange multipliers, duality results, scalarization of vector optimization problems, etc. The earliest version of the alternative theorem due to Fan, Glicksberg and Hoffman involves convex functions [6]. In order to generalize this version without convexity the authors in [9] introduce the notion of $*$-quasiconvexity with respect to the ordering cone $\mathbb{R}^m_+$ (it is called scalarly $\mathbb{R}^m_+$-quasiconvexity in [7]). Given a convex set $K \subseteq \mathbb{R}^n$, a function $F : K \to \mathbb{R}^m$, it said to be $*$-quasiconvex if

$$x \in K \mapsto \langle p^*, F(x) \rangle \text{ is quasiconvex for all } p^* \in \mathbb{R}^m_+. \tag{1.1}$$

By virtue of the preceding result in utility theory, one could expect the convexity of some components of f under $*$-quasiconvexity. Unfortunately, this is not the case as the following examples show, and even if semistrict quasiconvexity on each of the functions $x \mapsto \langle p^*, F(x) \rangle$ is imposed.

Example 1.1. *Let $K = [-1, +\infty[$. We consider $F : K \to \mathbb{R}^2$, $F = (f_1, f_2)$ with $f_1, f_2 : K \to \mathbb{R}$ defined by*

$$f_1(x) = \begin{cases} x^2, & \text{if } -1 \le x \le 1 \\ 1, & \text{if } x > 1, \end{cases}$$

$$f_2(x) = \begin{cases} x^4, & \text{if } -1 \le x \le 1 \\ 1, & \text{if } x > 1. \end{cases}$$

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Given $p = (p_1, p_2) \in \mathbb{R}^2_{++}$, *we have*

$$F_p(x) \doteq \langle p, F(x)\rangle = \begin{cases} p_1x^4 + p_2x^2, & \text{if } -1 \le x \le 1 \\ p_1 + p_2, & \text{if } x > 1. \end{cases}$$

We have that F_p *is quasiconvex for all* $(p_1, p_2) \in \mathbb{R}^2_{++}$, *but* f_1 *and* f_2 *are not convex.*

Example 1.2. *Let* $K = \mathbb{R}$. *We consider* $F : K \to \mathbb{R}^2$, $F = (f_1, f_2)$ *with* $f_1, f_2 : K \to \mathbb{R}$ *defined by*

$$f_1(x) = \begin{cases} -x & \text{if } x < -1 \\ x^2, & \text{if } -1 \le x \le 1 \\ x, & \text{if } x > 1, \end{cases}$$

$$f_2(x) = \begin{cases} -x & \text{if } x < -1 \\ x^4, & \text{if } -1 \le x \le 1 \\ x, & \text{if } x > 1. \end{cases}$$

Let $p = (p_1, p_2) \in \mathbb{R}^2_{++}$, *we obtain*

$$F_p(x) \doteq \langle p, F(x)\rangle = \begin{cases} -(p_1 + p_2)x & \text{if } x < -1 \\ p_1x^4 + p_2x^2, & \text{if } -1 \le x \le 1 \\ (p_1 + p_2)x & \text{if } x > 1. \end{cases}$$

We have that F_p *is quasiconvex and semistrictly quasiconvex for all* $(p_1, p_2) \in \mathbb{R}^2_{++}$, *but* f_1 *and* f_2 *are not convex.*

The purpose of this note is to provide sufficient conditions guaranteeing the quasiconvexity of two quasiconvex functions defined in reflexive Banach spaces. This will be done in an abstract framework by means of the characterization of quasiconvexity through the quasimonotonicity of an apropiate operator associated to the given functions. We will consider the subdifferential and the normal operators.

Throughout, X denotes a real Banach space, X^* its continuous dual and $\langle \cdot, \cdot \rangle$ the pairing between X and X^*. We will denote by $\mathbb{R}_{++}$ the set of strictly positive numbers, i.e., $\mathbb{R}_{++} =]0, +\infty[$.

Given a (single or set-valued) operator $T : X \rightrightarrows X^*$ its *graph* is the set

$$\mathrm{Gr}(T) = \big\{(x, x^*) \in X \times X^* : x^* \in T(x)\big\},$$

and its projection onto X is called the *domain* of T and will be denoted by $\mathrm{Dom}\, T$. A set-valued operator $T : X \rightrightarrows X^*$ is said to be *quasimonotone*, if for all $x_1, x_2 \in \mathrm{Dom}(T)$ and $x_1^* \in T(x_1)$,

$$\langle x_1^*, x_2 - x_1\rangle > 0 \Rightarrow \langle x_2^*, x_2 - x_1\rangle \ge 0, \text{ for all } x_2^* \in T(x_2).$$

Given a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ we denote by

$$\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$$

its domain (which we always assume nonempty), and for any $\lambda \in \mathbb{R}$

$$S_\lambda(f) = \{x \in X : f(x) \le \lambda\},$$

its *level set of order* λ, to simplify the notation we will write S_λ.

If f is differentiable we denote by

$$\mathrm{Crit}(f) = \{x \in \mathrm{dom}\, f' : f'(x) = 0\}.$$

its set of *critical points.*

An extended real-valued function $f : X \to \mathbb{R}\cup\{+\infty\}$ is called *quasiconvex,* if for all $x, y \in \operatorname{dom}(f)$

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\},\ \forall t \in]0,1[. \tag{1.2}$$

Equivalently, f is quasiconvex if and only if $S_\lambda(f)$ is a convex set for all $\lambda \in \mathbb{R}$. Recall that given a convex set C, *the normal cone* to C at $x \in \overline{C}$ is defined by

$$N_C(x) = \{x^* : \langle x^*, d\rangle \le 0, \forall d \in T_C(x)\},$$

where $T_C(x) = \overline{\bigcup_{\lambda>0} \lambda(C - \{x\})}$, the Bouligand tangent cone of C at $x \in \overline{C}$.

Many ways to characterize the quasiconvexity of a lower semicontinuous functions have been done in the literature, from which in connection to the generalized monotony we know two: via subdifferentials [1, 5, 8] and via cones normal to the level sets [3, 2].

1.1. **Subdifferential characterization.** Given a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, for $x \in \operatorname{dom} f$ we denote by

$$f^{\uparrow}(x,u) = \sup_{\delta>0} \limsup_{\substack{y \to_f x \\ t \searrow 0^+}} \inf_{v \in B(u,\delta)} \frac{f(y+tv) - f(y)}{t}$$

its Clarke-Rockafellar generalized derivative, where $t \searrow 0^+$ indicates the fact that $t > 0$ and $t \to 0$, and $y \to_f x$ means that both $y \to x$ and $f(y) \to f(x)$. Then the Clarke-Rockafellar subdifferential $\partial f(x)$ of the function f at the point x is defined as follows (cf. [4]):

$$\partial f(x) = \begin{cases} \{x^* \in X : f^{\uparrow}(x,u) \ge \langle x^*, u\rangle, \text{ for all } u \in X\}, & \text{if } x \in \operatorname{dom} f \\ \emptyset, & \text{otherwise.} \end{cases}$$

Theorem 1.3 ([1, Theorem 4.1])**.** *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then, f is quasiconvex if and only if ∂f is quasimonotone.*

1.2. **Normal characterization.** Given a lower semicontinuous function $f : X \to \mathbb{R}$, the set-valued operator $N_f : X \rightrightarrows X^*$ defined by

$$N_f(x) = \begin{cases} N_{S_{f(x)}}(x), & \text{if } x \in \operatorname{dom} f \\ \emptyset, & \text{otherwise,} \end{cases}$$

is the *Normal Operator* associated to f.

Theorem 1.4 ([2, Theorem 3,3])**.** *Let X a reflexive Banach space, and $f : X \to \mathbb{R}$ a lower semicontinuous function. Then, f is quasiconvex if and only if N_f is quasimonotone.*

2. MAIN RESULT

Proposition 2.1. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two quasimonotone operators. If for any $x, y \in \operatorname{Dom}(T_1 + T_2)$*

$$\mathbb{R}_{++}T_1(x) \subset \mathbb{R}_{++}T_2(x),$$

then $T_1 + T_2$ is a quasimonotone operator.

Proof. Let $(x, x^*) \in \mathrm{Gr}(T_1 + T_2)$ and $y \in \mathrm{Dom}(T_1 + T_2)$ such that

$$\langle x^*, y - x\rangle > 0. \tag{2.3}$$

Since $x^* = x_1^* + x_2^*$ with $x_1^* \in T_1(x)$ and $x_2^* \in T_2(x)$, we have that

$$\langle x_1^*, y - x\rangle > 0 \quad \text{or} \quad \langle x_2^*, y - x\rangle > 0.$$

Assume that

$$\langle x_1^*, y - x\rangle > 0. \tag{2.4}$$

By hypothesis there exists $\lambda > 0$ and $z^* \in T_2(x)$ such that $x_1^* = \lambda z^*$. Thus, we also have

$$\langle z^*, y - x\rangle > 0. \tag{2.5}$$

Using (2.4), (2.5) and the quasimonotony of T_1 and T_2 respectively, we get

$$\langle y_1^*, y - x\rangle \geq 0, \text{ for all } y_1^* \in T_1(y),$$

and

$$\langle y_2^*, y - x\rangle \geq 0, \text{ for all } y_2^* \in T_2(y).$$

Summing up these two inequalities, we get

$$\langle y_1^* + y_2^*, y - x\rangle \geq 0, \text{ for all } y_i^* \in T_i(y), \quad i = 1, 2,$$

or equivalently

$$\langle y^*, y - x\rangle \geq 0, \text{ for all } y^* \in (T_1 + T_2)(y).$$

The case $\langle x_2^*, y - x\rangle > 0$ is similar. □

In order to treat both kinds of characterization of lower semicontinuous quasiconvex functions, the subdifferential and the normal, we introduce the following notation: for a lower semicontinous function $f : X \to \mathbb{R} \cup \{+\infty\}$ we denote by $T_f : X \rightrightarrows X^*$ the operator which characterizes f in the sense that

$$f \text{ is quasiconvex, if and only if } T_f \text{ is quasimonotone.}$$

Theorem 2.2. *Let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be two lower semicontinuous quasiconvex functions. Assume that*

(1) for any $x, y \in \mathrm{Dom}(T_{f_1} + T_{f_2})$, $\mathbb{R}_{++}T_{f_1}(x) = \mathbb{R}_{++}T_{f_2}(x)$;
(2) $T_{f_1+f_2} = T_{f_1} + T_{f_2}$.

Then $f_1 + f_2$ is a quasiconvex functions.

Proof. Applying Proposition 2.1 we have that $T_{f_1} + T_{f_2}$ is a quasimonotone operator; the result follows from the definition of $T_{f_1+f_2}$. □

3. An application

Here, we exhibit an application of the notion of $*$-quasiconvexity. The result to be established is important by itself and ensures that any constrained scalar minimization problem can be reformulated with a single constrain under generalized convexity assumptions and a Slater-type condition. This result is a generalization of that obtained in [**?**]. Let us consider the following constrained minimization problem

$$\mu \doteq \inf_{x \in K} f(x), \tag{3.6}$$

where $K \doteq \{x \in C : \ g(x) \in -P\}$, C is a nonempty subset of a real locally convex topological vector space X, $f : C \to \mathbb{R}$, and $g : C \to Y$, with Y as before and $P \subseteq Y$ is a closed convex cone with nonempty interior. Let us consider also the assumption

(3.7)

$\forall p^* \in P^*$, the restriction of $\langle p^*, g(\cdot)\rangle$ on any line segment of K is continuous.

Theorem 3.1. *Let us consider problem (3.6) with f being quasiconvex. Assume that μ is finite and $g : C \to Y$ is $*$-quasiconvex such that for all $p^* \in P^* \setminus \{0\}$, $x \in C \mapsto \langle p^*, g(x)\rangle$ is semistrictly quasiconvex. If, in addition, the Slater-type condition that for some $\bar{x} \in C$, $\langle y^*, g(\bar{x})\rangle < 0$ for all $y^* \in P^* \setminus \{0\}$ holds. Then, there exists $p^* \in P^* \setminus \{0\}$ such that*

$$\inf_{\substack{g(x) \in -P \\ x \in C}} f(x) = \inf_{\substack{\langle p^*, g(x)\rangle \leq 0 \\ x \in C}} f(x) \tag{3.8}$$

Proof. Let us consider

$$M \doteq g(C_0) + P, \ C_0 \doteq \{x \in C : \ f(x) < \mu\}.$$

Since C_0 is convex and g is $*$-quasiconvex on any convex subset C' of C, the set M is convex by a corollary in [**?**]. We can assume that M is nonempty since otherwise any $p^* \in P^*$ verifies (3.8). Evidently, $M \cap (-P) = \emptyset$, for if not, there exists $z_0 \in -P$ such that $z_0 \in M$, that is, there is $x_0 \in C_0$ satisfying $z_0 - g(x_0) \in P$. It turns out that $g(x_0) \in -P$, $x_0 \in C$, $f(x_0) < \mu$, which cannot happen. We apply a convex separation theorem to obtain the existence of $p \in P^*$, $p^* \neq 0$, $\alpha \in \mathbb{R}$, such that

$$\langle p^*, z\rangle \geq \alpha \text{ for all } z \in M, \ \langle p^*, u\rangle \leq \alpha \text{ for all } u \in -P.$$

Hence,

$$p^* \in P^* \text{ and } \langle p^*, g(x)\rangle \geq 0 \text{ for all } x \in C_0. \tag{3.9}$$

Let $x \in C$, $\langle p^*, g(x)\rangle \leq 0$. In case $f(x) < \mu$, that is, $x \in C_0$, we get $g(x) \in M$ and thus $\langle p^*, g(x)\rangle = 0$. Set $x_t = t\bar{x} + (1-t)x$. Assume that $f(x_t) < \mu$ for some $t \in \,]0, 1[$, then by semistrict quasiconvexity, $0 \leq \langle p^*, g(x_t)\rangle < 0$, a contradiction. Whence $f(x_t) \geq \mu$ for all $t \in \,]0, 1[$. By continuity, $f(x) \geq \mu$. This implies

$$\inf_{\substack{\langle p^*, g(x)\rangle \leq 0 \\ x \in C}} f(x) \geq \inf_{\substack{g(x) \in -P \\ x \in C}} f(x).$$

The reverse inequality is trivial. $\square$

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