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Abstract

The purpose of this paper is to give an alternative proof of the Riesz representation theorem using the well-known theorem of Ky Fan minimax inequality applied to equilibrium problems.

Key words: Equilibrium problem, Convexity, Monotonicity, Coercivity.

1 Introduction

Let $H$ be a Hilbert space, and let $H'$ denote its dual space, consisting of all continuous linear functionals from $H$ into the field $\mathbb{R}$. It is very well known that for each $x \in H$, the function $x^* : H \to \mathbb{R}$ defined by

$$x^*(y) = \langle x, y \rangle,$$

for all $y \in H$

where $\langle \cdot, \cdot \rangle$ denote the inner product of $H$, is an element to $H'$. The Riesz Representation Theorem states that every element of $H'$ can be written uniquely in this form (see for instance [2]).

On the other hand, Blum and Oettli introduced the equilibrium problems in 1993 (see [1]), as a generalization of various problems such as minimization problems, Nash equilibrium, variational inequalities, etc (see for instance [1, 3]).

Formally, an equilibrium problem, associated to $K \subset H$ and $f : K \times K \to \mathbb{R}$, consists on finding $x \in K$ such that

$$f(x, y) \geq 0, \text{ for all } y \in K.$$

The solution set of an equilibrium problem is denoted by $EP(f, K)$.

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2 Preliminaries

Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. Let $K$ be a convex subset of $H$. Recall that a function $h : K \rightarrow \mathbb{R}$ is said to be:

- **convex** when for all $u, v \in K$ and all $t \in [0, 1]$ the following holds
  \[ h(tu + (1-t)v) \leq th(u) + (1-t)h(v) \]

- **quasiconvex** when for all $u, v \in K$ and all $t \in [0, 1]$ the following holds
  \[ h(tu + (1-t)v) \leq \max\{h(u), h(v)\} \]

- **lower semicontinuous** when for each $x_0 \in K$ and each $\lambda < f(x_0)$ there exists $\delta > 0$ such that for all $x \in K$ the following implication holds
  \[ |x - x_0| < \delta \Rightarrow f(x) > \lambda. \]

Finally, a function $f$ is called **concave**, **quasiconcave** or **upper semicontinuous** if $-f$ is convex, quasiconvex or lower semicontinuous, respectively.

Clearly, all convex functions are quasiconvex. Additionally, a function is continuous if, and only if, it is lower and upper semicontinuous.

A function $f : K \times K \rightarrow \mathbb{R}$, defined on $K \subset H$, is called:

- **monotone** when $f(u, v) + f(v, u) \leq 0$, for all $u, v \in K$;

- **coercive** when for all sequence $(u_n) \subset K$ with $|u_n| \rightarrow +\infty$ there exists $u \in K$ such that $f(u_n, u) \leq 0$, for all $n$ large enough.

Usually, the function $f$ is called **bifunction**.

The following result is due to Ky Fan who proved the famous **minimax inequality**.

**Theorem 2.1** ([4], Theorem 1). Let $V$ be a real Hausdorff topological vector space and $K$ a nonempty compact convex subset of $V$. If a bifunction $f : K \times K \rightarrow \mathbb{R}$ satisfies:

- $f(\cdot, y) : K \rightarrow \mathbb{R}$ is upper semicontinuous for each $y \in K$,
- $f(x, \cdot) : K \rightarrow \mathbb{R}$ is quasiconvex for each $x \in K$,

then there exists a point $x \in K$ such that

\[ \inf_{y \in K} f(x, y) \geq \inf_{w \in K} f(w, w). \]

The above theorem plays an important role in equilibrium problems, because Ky Fan’s Theorem implies existence of solutions for these problems.
3 An alternative proof of Riesz Representation Theorem

For each $\phi \in H'$, we define the bifunction $f : H \times H \to \mathbb{R}$ as

$$f(u, v) = \phi(u - v) - \langle u, u - v \rangle \text{ for all } u, v \in H$$

(R)

We note that $f$ satisfies the following property:

(i) $f(u, u) = 0$, for all $u \in H$.

(ii) Since $\phi$ and $\langle \cdot, \cdot \rangle$ are both continuous, we have that $f$ is continuous.

(iii) For all $u, v \in H$,

$$f(u, v) + f(v, u) = \phi(u - v) - \langle u, u - v \rangle + \phi(v - u) - \langle v, v - u \rangle = -|u - v|^2 \leq 0$$

thus, $f$ is monotone.

(iv) Let $u \in H$ we note that

$$f(u, \cdot) = \phi(u - \cdot) - \langle u, u - \cdot \rangle.$$ 

therefore, the function $f(u, \cdot)$ is affine.

(v) For each $v \in H$,

$$f(\cdot, v) = \phi(\cdot - v) - |\cdot|^2 + \langle \cdot, v \rangle$$

The linearity of $\phi$ and $\langle \cdot, v \rangle$, and concavity of $-|\cdot|^2$ imply that $f(\cdot, v)$ is concave.

**Theorem 3.1** (Riesz Representation Theorem). For each $\phi \in H'$, there exists an unique $u_0 \in H$ such that

$$\phi(u) = \langle u_0, u \rangle \text{ for all } u \in H.$$ 

Moreover, $|u_0| = |\phi|_{H'}$.

In order to proof the above proposition we need the following lemmas.

**Lemma 3.2.** Let $f$ defined as (R). If $u_1, u_2 \in EP(f, H)$ then $u_1 = u_2$.

**Proof.** The monotonicity of $f$ implies $f(u_1, u_2) = f(u_2, u_1) = 0$. So,

$$0 = f(u_1, u_2) = f(u_2, u_1) = |u_1 - u_2|^2$$

Therefore $u_1 = u_2$. 


Proof of Theorem 3.1. The uniqueness follows from Lemma 3.2.

For the existence, we consider $K = \overline{B}(0, |\phi|_{H'})$ which is weakly compact and convex. Since the bifunction $R$ satisfies the conditions of Theorem 2.1 on $K$, there exists $u_0 \in K$ such that
\[
\inf_{v \in K} f(u_0, v) \geq \inf_{w \in K} f(w, w) = 0
\]
i.e. $f(u_0, v) \geq 0$ for all $v \in K$ and in particular $f(u_0, 0) = \phi(u_0) - |u_0|^2 \geq 0$. We want to show that $f(u_0, \cdot)$ is linear. Since $f(u, \cdot)$ is affine it is enough to show that $f(u_0, 0) = \phi(u_0) - |u|^2 = 0$. Suppose that $|u_0|^2 < \phi(u_0)$, then
\[
0 \leq |u_0|^2 < \phi(u_0) = \|\phi(u_0)\| \leq \|\phi\|_{H'} |u_0| \quad \Rightarrow \quad |u_0| < |\phi|_{H'}
\]
and so there exists $t > 1$ such that $tu_0 \in K$. Thus,
\[
f(u_0, tu_0) = (1-t)\phi(u_0) - (1-t)|u_0|^2 = (1-t)[\phi(u_0) - |u_0|^2] \geq 0
\]
and this implies $\phi(u_0) - |u_0|^2 \leq 0$ which is a contradiction. Therefore, we have $\phi(u_0) = |u_0|^2$ and $f(u_0, \cdot)$ is linear.

So, for each $v \notin K$ there exists $t \in ]0, 1[$ such that $tv \in K$. The linearity of $f(u_0, \cdot)$ implies $f(u_0, v) = tv^{-1} \times f(u_0, tv) \geq 0$. Thus, $u_0 \in EP(f, H)$. Also by linearity of $f(u_0, \cdot)$ we have $f(u_0, -v) \geq 0$ which is true if and only if $f(u_0, v) \leq 0$ and therefore
\[
f(u_0, v) = \phi(u_0 - v) - \langle u_0, u_0 - v \rangle = 0, \quad \text{for all} \ v \in H
\]
which is equivalent to $\phi(w) = \langle u_0, w \rangle$, for all $w \in H$.

Finally, we note that
\[
\|\phi\|_{H'} = \sup_{u \in H, \|u\| \leq 1} \phi(u) = \sup_{u \in H, \|u\| \leq 1} \langle u_0, u \rangle \leq |u_0| \leq \|\phi\|_{H'}
\]
therefore $\|\phi\|_{H'} = |u_0|$. □

References


