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Abstract

The purpose of this paper is to give an alternative proof of the Riesz representation theorem using the well-known theorem of Ky Fan minimax inequality applied to equilibrium problems.

Key words: Equilibrium problem, Convexity, Monotonicity, Coercivity.

1 Introduction

Let H be a Hilbert space, and let H' denote its dual space, consisting of all continuous linear functionals from H into the field $\mathbb{R}$. It is very well known that for each $x \in H$, the function $x^* : H \to \mathbb{R}$ defined by

$$x^*(y) = \langle x, y \rangle, \text{ for all } y \in H$$

where $\langle \cdot, \cdot \rangle$ denote the inner product of H, is an element to H'. The Riesz Representation Theorem states that every element of H' can be written uniquely in this form (see for instance [2]).

On the other hand, Blum and Oettli introduced the *equilibrium problems* in 1993 (see [1]), as a generalization of various problems such as minimization problems, Nash equilibrium, variational inequalities, etc (see for instance [1, 3]).

Formally, an equilibrium problem, associated to $K \subset H$ and $f : K \times K \to \mathbb{R}$, consists on finding $x \in K$ such that

$$f(x, y) \geqslant 0, \text{ for all } y \in K.$$

The solution set of an equilibrium problem is denoted by $EP(f, K)$.

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2 Preliminaries

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let K be a convex subset of H. Recall that a function $h : K \to \mathbb{R}$ is said to be:

- *convex* when for all $u, v \in K$ and all $t \in]0, 1[$ the following holds

$$h(tu + (1 - t)v) \leqslant th(u) + (1 - t)h(v)$$

- *quasiconvex* when for all $u, v \in K$ and all $t \in]0, 1[$ the following holds

$$h(tu + (1 - t)v) \leqslant \max\{h(u), h(v)\}$$

- *lower semicontinuous* when for each $x_0 \in K$ and each $\lambda < f(x_0)$ there exists $\delta > 0$ such that for all $x \in K$ the following implication holds

$$\|x - x_0\| < \delta \;\Rightarrow\; f(x) > \lambda.$$

Finally, a function f is called *concave*, *quasiconcave* or *upper semicontinuous* if $-f$ is convex, quasiconvex or lower semicontinuous, respectively.

Clearly, all convex functions are quasiconvex. Additionally, a function is continuous if, and only if, it is lower and upper semicontinuous.

A function $f : K \times K \to \mathbb{R}$, defined on $K \subset H$, is called:

- *monotone* when $f(u, v) + f(v, u) \leqslant 0$, for all $u, v \in K$;
- *coercive* when for all sequence $(u_m) \subset K$ with $\|u_m\| \to +\infty$ there exists $u \in K$ such that $f(u_n, u) \leqslant 0$, for all n large enough.

Usually, the function f is called *bifunction*.

The following result is due to Ky Fan who proved the famous *minimax inequality*.

Theorem 2.1 ([4], Theorem 1)**.** *Let V be a real Hausdorff topological vector space and K a nonempty compact convex subset of V. If a bifunction $f : K \times K \to \mathbb{R}$ satisfies:*

- *$f(\cdot, y) : K \to \mathbb{R}$ is upper semicontinuous for each $y \in K$,*
- *$f(x, \cdot) : K \to \mathbb{R}$ is quasiconvex for each $x \in K$,*

then there exists a point $x \in K$ such that

$$\inf_{y \in K} f(x, y) \geqslant \inf_{w \in K} f(w, w).$$

The above theorem plays an important rol in equilibrium problems, because Ky Fan's Theorem implies existence of solutions for these problems.

3 An altertive proof of Riesz Representation Theorem

For each $\phi \in H'$, we define the bifunction $f : H \times H \to \mathbb{R}$ as

$$f(u, v) = \phi(u - v) - \langle u, u - v\rangle \text{ for all } u, v \in H \tag{R}$$

We note that f satisfies the following property:

(i) $f(u, u) = 0$, for all $u \in H$.

(ii) Since ϕ and $\langle\cdot,\cdot\rangle$ are both continuous, we have that f is continuous.

(iii) For all $u, v \in H$,

$$f(u, v)+f(v, u) = \phi(u-v)-\langle u, u-v\rangle+\phi(v-u)-\langle v, v-u\rangle = -\|u-v\|^2 \leqslant 0$$

thus, f is monotone.

(iv) Let $u \in H$ we note that

$$f(u, \cdot) = \phi(u - \cdot) - \langle u, u - \cdot\rangle.$$

therefore, the function $f(u, \cdot)$ is affine.

(v) For each $v \in H$,

$$f(\cdot, v) = \phi(\cdot - v) - \|\cdot\|^2 + \langle\cdot, v\rangle$$

The linearity of ϕ and $\langle\cdot, v\rangle$, and concavity of $-\|\cdot\|^2$ imply that $f(\cdot, v)$ is concave.

Theorem 3.1 (Riesz Representation Theorem)**.** *For each $\phi \in H'$, there exists an unique $u_0 \in H$ such that*

$$\phi(u) = \langle u_0, u\rangle \quad \textit{for all } u \in H.$$

Moreover, $\|u_0\| = \|\phi\|_{H'}$.

In order to proof the above proposition we need the following lemmas.

Lemma 3.2. *Let f defined as (R). If $u_1, u_2 \in EP(f, H)$ then $u_1 = u_2$.*

Proof. The monotonicity of f implies $f(u_1, u_2) = f(u_2, u_1) = 0$. So,

$$0 = f(u_1, u_2) + f(u_2, u_1) = \|u_1 - u_2\|^2$$

Therefore $u_1 = u_2$. □

Proof of Theorem 3.1. The uniqueness follows from Lemma 3.2.

For the existence, we consider $K = \overline{B}(0, \|\phi\|_{H'})$ which is weakly compact and convex. Since the bifunction (R) satisfies the conditions of Theorem 2.1 on K, there exists $u_0 \in K$ such that

$$\inf_{v \in K} f(u_0, v) \geqslant \inf_{w \in K} f(w, w) = 0$$

i.e. $f(u_0, v) \geqslant 0$ for all $v \in K$ and in particular $f(u_0, 0) = \phi(u_0) - \|u_0\|^2 \geqslant 0$. We want to show that $f(u_0, \cdot)$ is linear. Since $f(u, \cdot)$ is affine it is enough to show that $f(u_0, 0) = \phi(u) - \|u\|^2 = 0$. Suppose that $\|u_0\|^2 < \phi(u_0)$, then

$$0 \leqslant \|u_0\|^2 < \phi(u_0) = |\phi(u_0)| \leqslant \|\phi\|_{H'}\|u_0\| \quad \rightarrow \quad \|u_0\| < \|\phi\|_{H'}$$

and so there exists $t > 1$ such that $tu_0 \in K$. Thus,

$$f(u_0, tu_0) = (1-t)\phi(u_0) - (1-t)\|u_0\|^2 = (1-t)[\phi(u_0) - \|u_0\|^2] \geqslant 0$$

and this implies $\phi(u_0) - \|u_0\|^2 \leqslant 0$ which is a contradiction. Therefore, we have $\phi(u_0) = \|u_0\|^2$ and $f(u_0, \cdot)$ is linear.

So, for each $v \notin K$ there exists $t \in]0, 1[$ such that $tv \in K$. The linearity of $f(u_0, \cdot)$ implies $f(u_0, v) = t^{-1} \times f(u_0, tv) \geqslant 0$. Thus, $u_0 \in EP(f, H)$. Also by linearity of $f(u_0, \cdot)$ we have $f(u_0, -v) \geqslant 0$ which is true if and only if $f(u_0, v) \leqslant 0$ and therefore

$$f(u_0, v) = \phi(u_0 - v) - \langle u_0, u_0 - v \rangle = 0, \quad \text{for all } v \in H$$

which is equivalent to $\phi(w) = \langle u_0, w \rangle$, for all $w \in H$.

Finally, we note that

$$\|\phi\|_{H'} = \sup_{u \in H,\ \|u\| \leqslant 1} \phi(u) = \sup_{u \in H,\ \|u\| \leqslant 1} \langle u_0, u \rangle \leqslant \|u_0\| \leqslant \|\phi\|_{H'}$$

therefore $\|\phi\|_{H'} = \|u_0\|$.

□

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